Section 4-1 and 4-2 Mathematics 108

Exponential Functions

In the future, you are interviewing at a super corporation called Apple-Google-Facebook. They are the best place in the world to work, with the best benefits and the highest salaries. All you need to get hired is to show how good you are at math by answering the following question correctly.

When they hire you, you have a choice of salaries.

Choice 1:

You start at a penny a day. This doubles every day for the first month and then stays the same from then on.

Choice 2:

You get a fixed salary of $1,000,000 a day.

You are guaranteed to be employed for just 1 month.

You have to choose quickly.

Let's do the calculation:

$2^{30} \sim 107,000,000$, and that's just the last day. you also get

53,000,000
26,000,000
13,000,000
6,000,000
3,000,000
1,000,000

More than 209 Million dollars, just from starting with 1 cent.

What makes this surprising is that we are dealing with an **Exponential** function.

Definition of an exponential function

$f(x) = a^x$ where $a > 0$ and $a \neq 1$

If $a > 1$ then f is an increasing exponential function.

Note that if $0 < a < 1$ then $\frac{1}{a} > 1$ and so

$f(x) = \left(\frac{1}{a}\right)^x = a^{-x}$

This is a decreasing exponential function.

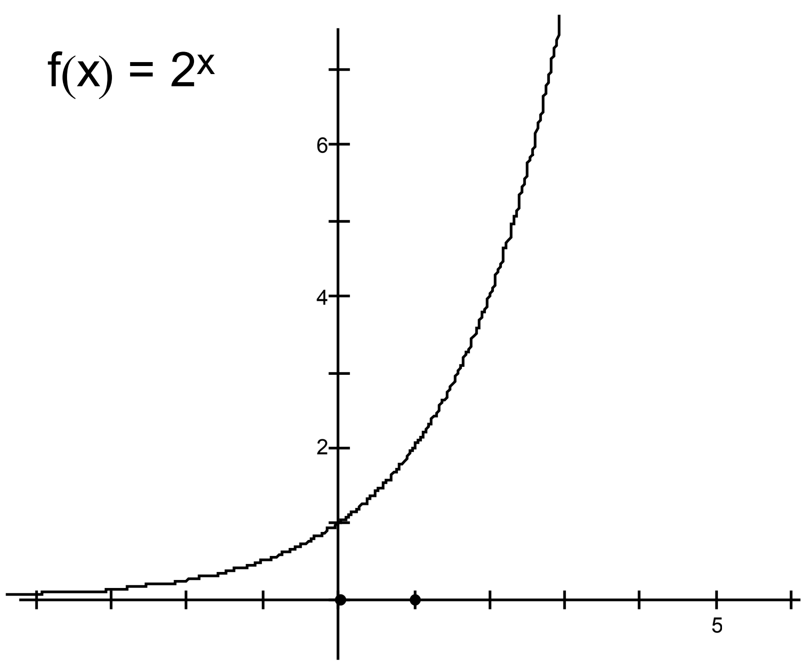
An increasing exponential functions has the property that it will eventually dominate any polynomial function.

That is if you have $E(x)$ and $P(x)$, no matter how you construct them, eg.

$E(x) = 1.000001^x$ and $P(x) = x^{1000,000,000}$

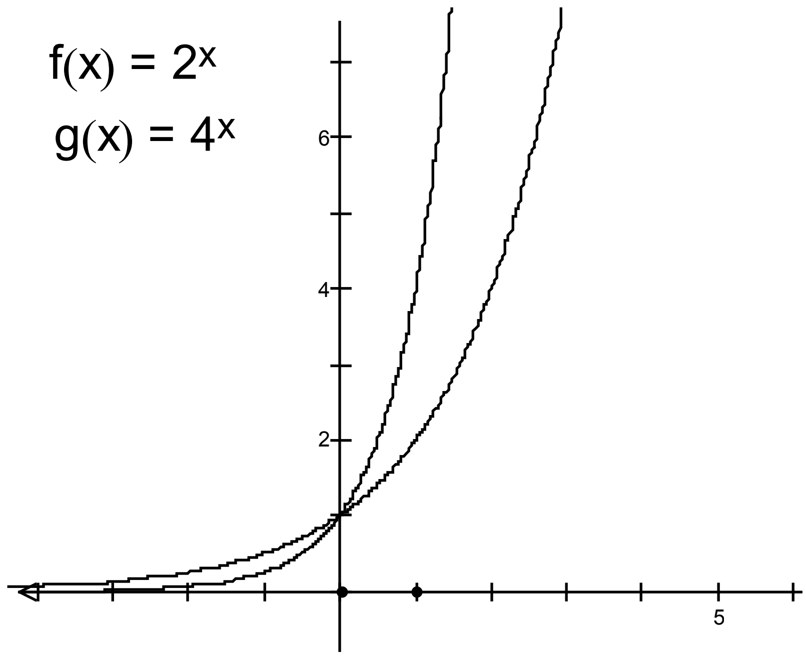
for some large C if $x > C$ then $E(x) > P(x)$

Let's look at a simple exponential function.



Note that it has the x axis as an asymptote and passes through the Y -intercept y=1.

Except for the rate at which the function rises or falls, the shape of all increasing exponential functions are the same.



$4^x = \left(2^2\right)^x = 2^{2x}$

So $g(x)$ is just a dilation transformation in the x direction.

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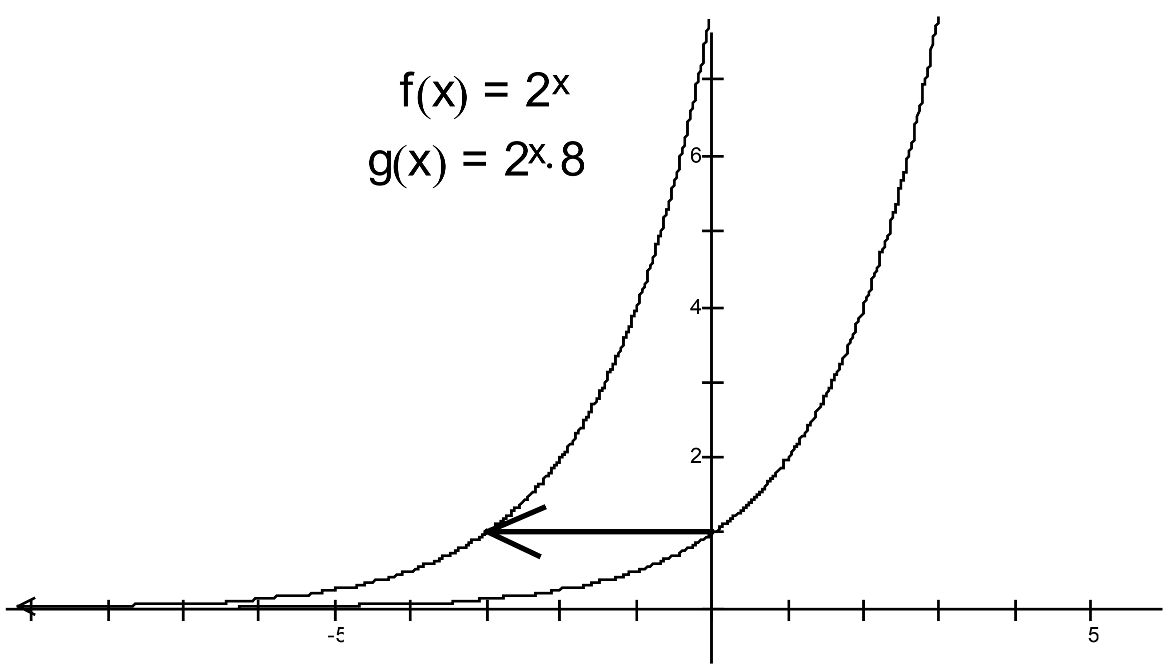
What does multiplying by a constant do?

$f\left(x\right) = 8 \cdot 2^x = 2^2 \cdot 2^x = 2^{x+2}$

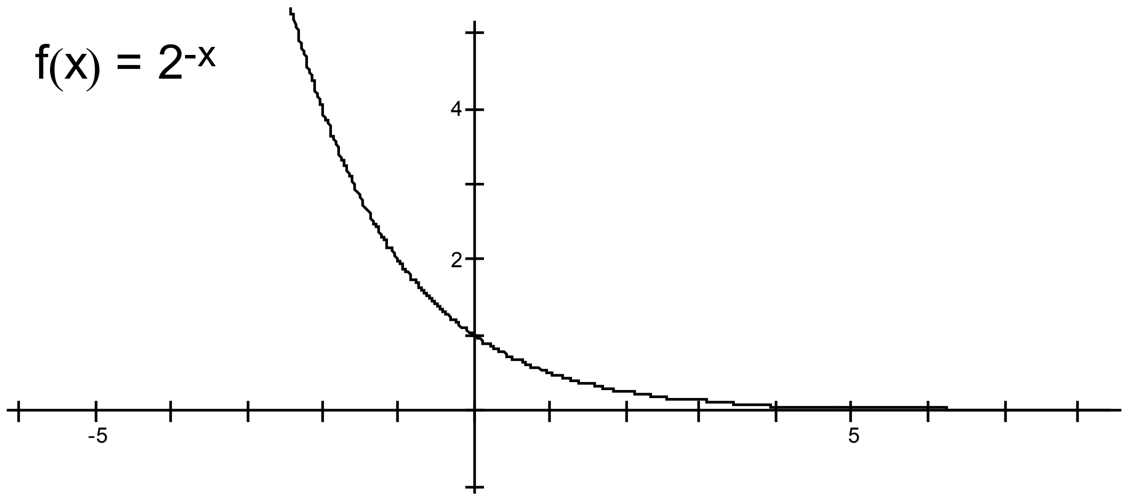
So it is just a shift transformation.



The exponential functions we've looked at are increasing functions, meaning that from left to right they always increase.

An exponential function can also be decreasing.

$$f(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$$



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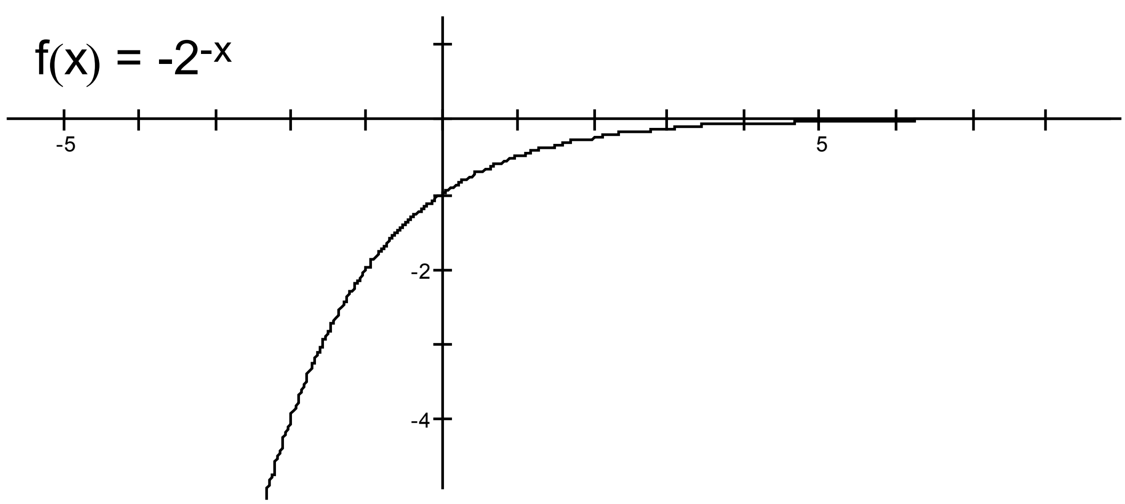
You might notice this is just a reflection transformation on the function, reflected through the *y* axis.

Sometimes expanding exponential functions are called growth functions, and decreasing functions are called decay functions.

As you might expect, multiplying the function by -1, just reflects it across the *Y*-axis.



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Modeling nuclear decay using an exponential function
or how to store cat litter until it's safe, :-).

Example:

THIS IS A REAL LIFE EXAMPLE THAT I HAD TO DEAL WITH

My cat had a condition called hyperthyroid.

His thyroid gland had a benign (non-cancerous) tumor that caused the thyroid gland to grow and overproduce thyroid hormones. This happens to people too, and the treatment is the same. The consequence of this is a cat will lose its appetite and not eat enough.

The treatment is seems very unusual at first. They inject the cat with a radioactive isotope of iodine called iodine 131. The way this works is that the thyroid gland filters iodine and stores it. So the radioactive iodine accumulates in the cat's thyroid gland. There it destroys the faster growing tumor.

While this is happening the cat is isolated because the radioactive rays could cause cancer in a person. After the cat was released, we had take special precautions with the cat litter. The problem is you can't put radioactive material into your garbage.

Radioactive materials are measured in how fast they decay by a half life. The half life of a material is the time it takes for half of the material to decay. This is a decreasing exponential function. After the first half is gone, it takes the same amount of time for half of what is left to decay.

I needed to store the cat litter until the amount of radioactive material was 1/1000,000 the original amount before I could throw it away.

The half life of iodine 131 is 8.02 days. How many days would I need to wait?

The amount of iodine left is

$$I(d) = I_0\left(\frac{1}{2}\right)^{\frac{d}{8.02}}$$

Rewriting this as $\dfrac{I(d)}{I_o} = \left(\dfrac{1}{2}\right)^{\frac{d}{8.02}}$

I want to find out when $\dfrac{I(d)}{I_o} = \left(\dfrac{1}{2}\right)^{\frac{d}{8.02}} = \dfrac{1}{1{,}000{,}000}$

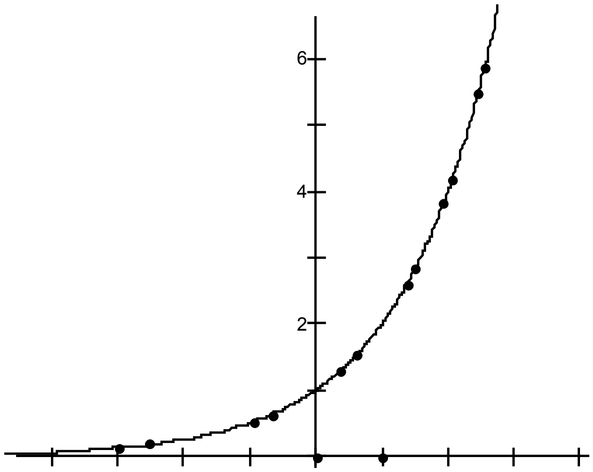
On Friday we will find a more direct way to solve this, but for now I will use trial and error.

d	$\frac{I(d)}{I_o}$ $\frac{I(d)}{I_o}$
10	.42
100	.00018
150	.0000023
160	.000001

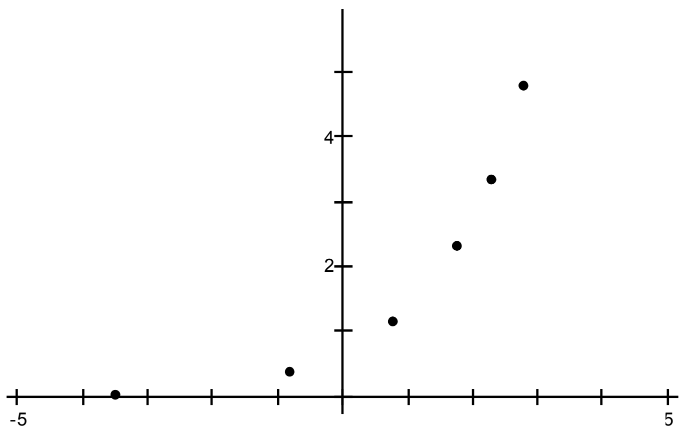
So about 160 days or about 5 months.

It sat outside in my backyard in a garbage can for about that long.

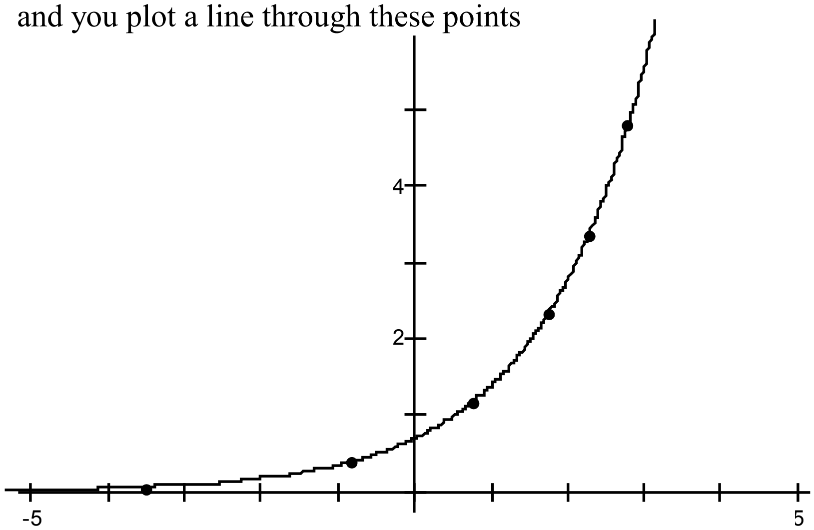
An interesting property of exponential functions



If you plot an exponential function and choose close points, and calculate the average rate of change between those points, and then you plot those points,



and you plot a line through these points



You get another exponential function!

Evaluating the value of an exponential function at a specific x value

One thing you might be concerned about has to do with evaluating an exponential function at a specific x value.

For all x values that are rational, this is not a problem.

Example:

$$2^{\frac{123}{454}}$$

We know from the laws of exponents that

$$2^{\frac{123}{454}} = \sqrt[454]{2^{123}}$$

It might be hard to evaluate this, but at least we know what the expression means.

This is not quite as clear when we have something like this:

$2^{\sqrt{3}}$ or 2^{π} These exponents are not rational. If we write them as a decimal expansion, they never end. On the other hand we can create a table of approximations.

Since $\sqrt{3} \sim 1.732050$

	$< 2^x$	$> 2^x$
1.7	$2^{1.7} = 3.2490...$	$2^{1.8} = 3.4822...$
1.73	$2^{1.73} = 3.3173...$	$2^{1.74} = 3.3403...$
1.732	$2^{1.732} = 3.3219...$	$2^{1.733} = 3.3241...$
1.7320	$2^{1.7320} = 3.3219...$	$2^{1.73201} = 3.3221...$
1.73205	$2^{1.73205} = 3.3220...$	$2^{1.73206} = 3.3220...$

So as we use better and better approximations for $\sqrt{3}$ our approximation of $2^{\sqrt{3}}$ gets squeezed between to values that get closer and closer together.

After learning calculus we will say that the value approaches a limit, and we assign the value of $f(x)$ that limit.

An Increasing exponential functions as a model for Compound Interest.

When you invest money in a bank, a money market fund or you buy a bond with a fixed interest rate, the bank, fund or bond pays you interest on some time period basis, called the **compounding period**.

Interest rates are almost always stated on a yearly basis, so when figuring out the interest you've earned you need to do a calculation because the compounding period could be yearly, monthly, daily, or constantly. After each compounding period

If for example you should be so lucky today to get 5% yearly interest on your money, compounded monthly.

Interest/month = $\frac{.05}{12}$

If you invest your original **principle** $\sum_{i=1}^{100} x$ for 1month, then the total amount of your investment P after one month is

$$P(1) = P_0\left(1+\frac{.05}{12}\right)$$

After two months it is

$$P(2) = P_0\left(1+\frac{.05}{12}\right)\left(1+\frac{.05}{12}\right)$$

and after m months it will be

$$P(m) = P_0\left(1+\frac{.05}{12}\right)^m$$

This of course is an exponential function. Remember what I said about exponential functions growing faster than any polynomial function.

Getting fixed interest is a great way to make money if you have enough time. Think of the fact that you will probably not retire for 45 years and consider how much any investments you make now will grow.

Effect of the compounding period

What effect does the compounding period have.

Example:

You invest $1000 for 10 years at an interest rate of 5% per year.
How much do you end up with if the compounding period is yearly, monthly, daily or even more often.

Compounded yearly

$$P(10) = 1000\left(1 + \frac{.05}{1}\right)^{10} = \$1628.89$$

Compounded monthly

$$P(10 \cdot 12) = 1000\left(1 + \frac{.05}{12}\right)^{10 \cdot 12} = \$1647.01$$

Compounded daily

$$P(10 \cdot 365) = 1000\left(1 + \frac{.05}{365}\right)^{10 \cdot 365} = \$1648.66$$

or more generally if n is the number of periods per year and r is the annual interest rate.

$$P(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}$$

The amazing Euler's constant *e*.

Think of this expression $\left(1+\frac{1}{n}\right)^n$

Consider what happens as n gets larger and larger

n	$\left(1+\frac{1}{n}\right)^n$
1	2
10	2.59
100	2.705
1000	2.7169
10000	2.7184
1000000	2.71828

As n gets larger and larger, the expression converges on an irrational value we call e or Euler's constant.

This number is probably only second in importance in mathematics to π.

If we do the same things with the expression $\left(1+\frac{x}{n}\right)^n$

we find it approaches the value e^x

This is useful if we want to find the principle for a loan where we continuously compound the interest. We get tje eqiatopm/

$P(t)=P_0e^{rt}$ where r is the rate of interest.

Another amazing thing about this number is that if we look at the function

$f(x)=e^x$ and perform our average rate of change experiment on it, the new function is

$f(x)=e^x$

Another indication that this is a very important number, especially with respect to exponential functions is that if you look on any scientific calculator, you will see a special button $\left[e^x\right]$.